Published by Institute of Physics Publishing for SISSA/ISAS
Received: *June 7, 2004*
Accepted: *June 25, 2004*

Exact superpotentials, theories with flavor and confining vacua

Marta Gómez-Reino

Martin Fisher School of Physics, Brandeis University
Waltham, MA 02454, U.S.A.
E-mail: marta@brandeis.edu

Abstract: In this paper we study some interesting properties of the effective superpotential of $\mathcal{N}=1$ supersymmetric gauge theories with fundamental matter, with the help of the Dijkgraaf-Vafa proposal connecting supersymmetric gauge theories with matrix models. We find that the effective superpotential for theories with N_f fundamental flavors can be calculated in terms of quantities computed in the pure ($N_f = 0$) gauge theory. Using this property we compute in a remarkably simple way the exact effective superpotential of $\mathcal{N}=1$ supersymmetric theories with fundamental matter and gauge group $\mathrm{SU}(N_c)$, at the point in the moduli space where a maximal number of monopoles become massless (confining vacua). We extend the analysis to a generic point of the moduli space, and show how to compute the effective superpotential in this general case.

Keywords: Supersymmetric Effective Theories, Matrix Models.

http://jhep.sissa.it/archive/papers/jhep062004051/jhep062004051.pdf

Contents

1. Introduction

Over the past few years much progress has been made in computing effective superpotentials of $\mathcal{N}=1$ supersymmetric gauge theories. Motivated by geometric considerations of dualities in string theory [1, 2], an expression for the quantum effective superpotential was proposed by Dijkgraaf and Vafa. They conjectured that the effective superpotential can be calculated by doing perturbative computation in an auxiliary matrix model [3, 4, 5], being later proved with perturbative field theory arguments in [6] and by the analysis of the generalized Konishi anomaly in [7, 8, 9]. This proposal provides direct connections between the computations in the matrix model descriptions with those in supersymmetric gauge theories. The proposal was later extended to the addition of matter in the fundamental representation of the gauge group in [10]–[15].

In this paper we will be considering the $\mathcal{N}=1$ supersymmetric gauge theory with matter in the fundamental representation of the gauge group which can be obtained by deforming the $\mathcal{N}=2$ gauge theory via the addition of a tree level superpotential. Actually there has been significant work on using the Dijkgraaf-Vafa proposal to get results of $\mathcal{N}=2$ theories [14, 16]–[21]

In the case of $\mathcal{N}=1$ theories with fundamental matter the effective superpotential will have contributions coming from planar diagrams with one boundary, apart from the contribution coming from the planar diagrams with no boundaries [10]. As we will see in the first part of the paper, those contributions to the superpotential can be computed, within the matrix model setup, in terms of traces of certain matrix model operators. We find that those traces should be computed in the pure gauge theory, even to calculate the effective superpotential for theories with flavor. As a direct application of this we will compute the exact superpotential for a $\mathcal{N}=1$ theory with gauge group $\mathrm{SU}(N_c)$ and $N_f < N_c$ fundamental matter hypermultiplets at the point of the moduli space where a maximal number of monopoles become massless. From the point of view of the underlying Seiberg-Witten curve [22] this correspond to the point where the curve factorizes completely. The moduli that factorizes the Seiberg-Witten curve in the pure gauge theory (no matter) case are well known [23], and they will be the only ingredient that we need to compute the exact effective superpotential in this case, even though we are considering theories with fundamental matter. We will find that the result for the exact superpotential in this case is remarkably simple. We will also consider the generalization of these results to an arbitrary point of the moduli space with n distinct glueball superfields, by using the techniques developed in [7] to compute traces of operators within matrix models.

The plan of this paper is as follows: in section 1 we will review briefly the Dijkgraaf-Vafa proposal for $\mathcal{N}=1$ theories with $\mathrm{SU}(N_c)$ gauge group, and N_f matter hypermultiplets in the fundamental representation of the gauge group, and we will set up the ingredients that we will need for the calculation of the effective superpotential. In section 3 we will compute, using the results in section 2, the exact effective superpotential for $\mathcal{N}=1$ theories with fundamental matter in the point of the moduli space where a maximal number of monopoles become massless. In section 4 we will extend the analysis of section 3 to a generic point of the moduli space and show how to compute the effective superpotential in this general case. Finally, in section 5 we conclude with a summary of the work.

2. The effective superpotential of $\mathcal{N}=1$ supersymmetric gauge theories

In this paper we will be considering an $\mathrm{SU}(N_c)$ gauge theory with $N_f < N_c$ pairs of quark fields q_i ($\tilde{q}_i$), $i=1,\ldots,N_f$, in the fundamental (anti-fundamental) representation of $\mathrm{SU}(N_c)$. The lagrangian density of this theory is given by

$$\mathcal{L} = \int d^4\theta\, \mathrm{Tr}(\bar{q}_i \mathrm{e}^V q_i + \tilde{q}_i \mathrm{e}^V \bar{\tilde{q}}_i) + \int d^2\theta\, \mathrm{Tr}\left(W(\phi, q, \tilde{q}) + \tau \mathcal{W}^\alpha \mathcal{W}_\alpha\right), \tag{2.1}$$

where $\mathcal{W}_\alpha$ is the gauge superfield and the superpotential W is given by

$$W(\phi, q, \tilde{q}) = W(\phi) + \phi q \tilde{q} - q m \tilde{q}\,. \tag{2.2}$$

The first term of (2.2) is a polynomial tree level superpotential for the adjoint Higgs field ϕ and m is the mass matrix for the flavors. Classically at the critical points of this tree level superpotential the gauge group breaks to $\prod_k \mathrm{SU}(N_k)$ where N_k is the number of eigenvalues of the Higgs field ϕ at the k-th critical point. Quantum mechanically there will be a gluino condensate S_k for each of the factors and an effective superpotential for these condensates. In this section we will review the Dijkgraaf-Vafa approach to the computation of the effective superpotential for these condensates as a planar limit of a given matrix models [3, 4, 5].

2.1 Theories with flavors from matrix models

According to the proposal given by Dijkgraaf and Vafa in [3, 4, 5] the effective superpotential of $\mathcal{N}=1$ gauge theories obtained as a deformation of $\mathcal{N}=2$ theories by an arbitrary tree level superpotential can be computed using matrix models. In the generalization of this proposal to include fields in the fundamental representation of the gauge group (this case was first considered in [10]) this superpotential is given by

$$W_{\text{eff}}(S) = N_c \frac{\partial \mathcal{F}_{\chi=2}}{\partial S} + \mathcal{F}_{\chi=1}\,, \tag{2.3}$$

where the $\mathcal{F}_{\chi=1,2}$ are defined through the matrix integral

$$Z = \mathrm{e}^{-\sum_\chi \frac{1}{g_s^\chi}\mathcal{F}_\chi} = \frac{1}{\mathrm{Vol}(G)} \int \mathcal{D}\Phi \mathcal{D}q \mathcal{D}\tilde{q}\, \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr} W(\Phi,q,\tilde{q})}\,, \tag{2.4}$$

and the superpotential $W(\Phi,q,\tilde{q})$ is given by eq. (2.2), but replacing the gauge theory fields by matrices $(\phi,q,\tilde{q}) \to (\Phi,Q,\tilde{Q})$. Following the approach in [15] for the generalization of the Dijkgraaf-Vafa proposal to theories with fundamental matter, Φ will be here a $M\times M$ matrix, Q a $M\times M_f$ matrix and $\tilde{Q}$ a $M_f\times M$ matrix, where the parameters M, M_f are unrelated to the gauge theory parameters N_c, N_f.

Then, introducing the parameters $S\equiv g_s M$ and $S_f \equiv g_s M_f$ the dependence of the free energy on the quantities g_s, M, and M_f can easily be extracted from the topology of the diagrams, and can be written as an expansion in the genus g and the number of quark loops h as

$$\mathcal{F} = \sum_{g,h} g_s^{2g-2} S_f^h \mathcal{F}_{g,h}(S)\,. \tag{2.5}$$

The planar contributions can be found from a large M expansion of the matrix model. All planar diagrams are summed by taking both the rank of the gauge group and the number of flavors to infinity (M, $M_f\to\infty$) and ($g_s\to 0$), while keeping $S=g_sM$ and $S_f=g_sM_f$ finite. From the expansion in (2.5) we see that this limit picks out the genus zero contribution, and also that to reproduce the effective superpotential (2.3) one should consider at most one quark loop and set S_f to zero at the end of the calculation (see [15] for details)

$$W_{\text{eff}}(S) = N_c \frac{\partial \mathcal{F}}{\partial S}\bigg|_{S_f=0} + N_f \frac{\partial \mathcal{F}}{\partial S_f}\bigg|_{S_f=0} \equiv N_c \frac{\partial \mathcal{F}_{\chi=2}}{\partial S} + \mathcal{F}_{\chi=1}\,. \tag{2.6}$$

Furthermore, as the matter fields in (2.4) appear only quadratically, we can integrate them out. This generates a $\log(\Phi+m)$ potential for Φ and we are then left with an integral just over Φ [11, 14, 15]

$$Z = \mathrm{e}^{-\mathcal{F}} = \int \mathcal{D}\Phi\, \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr}\tilde{W}(\Phi)}\,, \tag{2.7}$$

where

$$\tilde{W}(\Phi) = W(\Phi) + S_f \sum_{k=1}^{N_f} \log(\Phi - m_k)\,, \tag{2.8}$$

and $W(\Phi)$ is an arbitrary tree level superpotential

$$W(\Phi) = \sum_{p\geq 1} g_p \frac{\mathrm{tr}\Phi^p}{p} \,. \tag{2.9}$$

In the following subsection we will show how to deduce some very interesting properties of $\mathcal{F}_{\chi=2}$ and $\mathcal{F}_{\chi=1}$ with the help of (2.5), (2.7) and (2.8). Those properties will be actually useful to compute the effective superpotential exactly in several cases.

2.2 Properties of the $\mathcal{F}_{\chi=2}$ and $\mathcal{F}_{\chi=1}$ contributions to the superpotential

The $\mathcal{F}_{\chi=2}$ and $\mathcal{F}_{\chi=1}$ contributions to the effective superpotential can be evaluated perturbatively about an extremal point of the tree level superpotential [16, 20, 14]. From the perturbative expansion it is easy to see that the form of $\mathcal{F}_{\chi=2}$ in terms of the glueball superfield remains unchanged when including fundamental matter in the theory. All the explicit dependence in the matter of the hypermultiplets appears in the $\mathcal{F}_{\chi=1}$ contribution.

$\chi = 2$ contribution. Taking derivatives of the free energy with respect to the parameter g_s has been proven to be a useful tool to get non-trivial information about the effective superpotential in the pure gauge group (no matter) case [24]. We will check here that the relation obtained in [24] (see also [25]) remains unchanged with the addition of matter and we will also see how to use that relation to compute the $\mathcal{F}_{\chi=2}$ contribution to the superpotential.

From eq. (2.5) we can calculate the derivative of the free energy with respect to g_s within our setup

$$\partial_{g_s}\mathcal{F} = \sum_{g,h} g_s^{2g-3}(S_f)^h \left((2g-2)\mathcal{F}_{g,h} + S\frac{\partial \mathcal{F}_{g,h}}{\partial S}\right) + \sum_{g,h} h g_s^{2g-3}(S_f)^h \mathcal{F}_{g,h}(S) \,. \tag{2.10}$$

As we already mentioned in the previous section the genus zero contribution is obtained by taking the $g_s = 0$ limit and by considering at most one quark loop ($S_f = 0$). Therefore we get that in this limit

$$\partial_{g_s}\mathcal{F} = g_s^{-3}\left(S\frac{\partial \mathcal{F}_{\chi=2}}{\partial S} - 2\mathcal{F}_{\chi=2}\right) \,. \tag{2.11}$$

Note that the relation (2.11) does not involve any explicit contribution from the matter of the hypermultiplets, as it does not depend on $\mathcal{F}_{\chi=1}$.

On the other hand, since in the planar limit the partition function Z of the matrix model is given by (2.7), the free energy is

$$\mathcal{F} = -\log Z = -\log\left(\int d\Phi e^{-g_s^{-1}\mathrm{tr}\tilde{W}(\Phi)}\right) \,. \tag{2.12}$$

From the definition of $\mathcal{F}$ as a free energy we know that its derivative over a parameter is a vacuum expectation value of the correspondingly coupled operator. Therefore

$$\partial_{g_s}\mathcal{F} = -g_s^{-2}\langle \mathrm{tr}\tilde{W}(\Phi)\rangle + g_s^{-2} S_f \sum_{k=1}^{N_f} \langle \mathrm{tr}\log(\Phi + m_k)\rangle = -g_s^{-2}\langle \mathrm{tr} W(\Phi)\rangle \,, \tag{2.13}$$

JHEP06(2004)051

where we have used the definition of $\mathrm{tr}\tilde{W}(\Phi)$ given by (2.8). Remember that, as we already mentioned, we are interested on the limit that takes into account just the genus zero contribution (that is, $g_s = 0$ and $S_f = 0$). This means that, once we take that limit, the vacuum expectation value in (2.13) has to be computed as in the pure $\mathrm{SU}(N_c)$ theory. Then, comparing (2.11) and (2.13) we get that

$$g_s \langle \mathrm{tr} W(\Phi) \rangle_0 = 2\mathcal{F}_{\chi=2} - S \frac{\partial \mathcal{F}_{\chi=2}}{\partial S} \,, \tag{2.14}$$

where by $\langle \ldots \rangle_0$ we mean vacuum expectation value in the pure $\mathrm{SU}(N_c)$ theory. The relation (2.14) is exactly the same as the one found in [24] for a gauge theory without matter content, and it is therefore a check that the form of $\mathcal{F}_{\chi=2}$ in terms of the glueball superfield S remains unchanged with the addition of matter. Also the fact that within matrix models there is a well developed technique to compute vacuum expectation values [7] implies that eq. (2.14) can be used to compute $\mathcal{F}_{\chi=2}$ to all orders in perturbation theory. We will show this explicitly in the following sections.

$\chi = 1$ **contribution.** As we mentioned in the beginning of this section, all the matter dependence that will appear on the effective superpotential will be encoded on the $\mathcal{F}_{\chi=1}$ contribution. As in the previous case, we would like to find an expression for $\mathcal{F}_{\chi=1}$ involving vacuum expectation values, but taking derivatives of the free energy with respect to the parameter g_s does not give us any information about $\mathcal{F}_{\chi=1}$. Nevertheless in our case, contrary to the pure case, we still have another parameter in the theory: S_f. If we take a derivative of the free energy $\mathcal{F}$ in (2.5) with respect to S_f we get

$$\partial_{S_f} \mathcal{F} = \sum_{g,h} h g_s^{2g-2} (S_f)^{h-1} \mathcal{F}_{g,h}(S) \,. \tag{2.15}$$

Therefore the genus zero contribution ($g_s = 0$ and $S_f = 0$) will pick out the term $\partial_{S_f}\mathcal{F} = \frac{1}{g_s^2}\mathcal{F}_{\chi=1}$. When taking a derivative of (2.12) with respect to S_f we get

$$\partial_{S_f} \mathcal{F} = g_s^{-1} \sum_{k=1}^{N_f} \langle \mathrm{tr} \log(\Phi + m_k) \rangle \,. \tag{2.16}$$

Therefore in the genus zero limit we have that (by comparing eq. (2.15) and (2.16))

$$\mathcal{F}_{\chi=1} = g_s \sum_{k=1}^{N_f} \langle \mathrm{tr} \log(\Phi + m_k) \rangle_0 \,, \tag{2.17}$$

where the *vev* is again meant to be computed in terms of the pure $\mathrm{SU}(N_c)$ theory. We will show in the following sections how can (2.17) be used to calculate $\mathcal{F}_{\chi=1}$.

3. Exact superpotentials for theories with flavors in confining vacua

As a first application to the relations (2.14) and (2.17) obtained in the previous section we will compute in this section the exact effective superpotential of a certain $\mathcal{N} = 1$

supersymmetric theory. We will consider a $\mathcal{N}=1$ theory obtained by perturbing a $\mathcal{N}=2$ supersymmetric by a tree level superpotential $\mathrm{SU}(N_c)$ gauge theory with N_f matter hypermultiplets, in the point of the moduli space where N_c-1 monopoles become massless. The exact effective superpotential was computed in [17] for a pure gauge theory using the "integrating in" procedure. The addition of fundamental matter was considered in [10, 33] for the particular case of a quadratic tree level superpotential, $W_{\text{tree}}=\frac{1}{2}\mathrm{tr}\Phi^2$, and in [30], using random matrix models, for a more general tree level superpotential. The method described in this section shows that in order to obtain the exact superpotential with fundamental matter in the maximally degenerating point of the moduli space we just need to compute, within the matrix model formulation, vacuum expectation values of several operators. This is a considerable advantage compared to previous works. As we will see, those vacuum expectation values are very easy to compute in the present case, due to the fact that we just need information about the pure gauge theory, so the actual computation of the exact effective superpotential turns out to be very simple.

3.1 Exact superpotentials for $\mathrm{SU}(N_c)$ theories in confining vacua

The quantum moduli space of $\mathcal{N}=2$ supersymmetric $\mathrm{SU}(N_c)$ gauge theories with N_f massive flavors is (N_c-1)-dimensional, and it is parametrized by the moduli u_k, $k=2,\dots,N_c$. At each point of the moduli space, the low energy theory is described by an $\mathcal{N}=2$ effective theory where the gauge group is broken to $\mathrm{U}(1)^{N_c-1}$. All the information about hte $\mathcal{N}=2$ theory is encoded in a particular meromorphic one-form $d\lambda_{SW}$ defined over an auxiliary curve, the Seiberg-Witten curve [22, 35]

$$y^2=P_{N_c}(x,u_k)^2-4\Lambda^{2N_c-N_f}\prod_{f=1}^{N_f}(x+m_f)\,, \tag{3.1}$$

where $P_{N_c}(x,u_k)$ is the characteristic polynomial of $\mathrm{SU}(N_c)$ that is given by

$$P_{N_c}(x,u_k)=x^{N_x}-\sum_{k=2}^{N_c}u_k x^{N_c-k}\,. \tag{3.2}$$

In this section we will be interested in the case where N_c-1 mutually local monopoles condense. This corresponds to a complete factorization of the Seiberg-Witten curve

$$P_{N_c}(x,u_k)^2-4\Lambda^{2N_c-N_f}\prod_{i=1}^{N_f}(x+m_i)=(x-x_1)(x-x_2)H^2_{N_c-1}(x)\,. \tag{3.3}$$

For the case of pure $\mathrm{SU}(N_c)$ the solution to this problem was found in [23] with the help of Chebyshev polynomials. The moduli that factorize the curve in the pure gauge theory case are given by

$$u^0_{2p}=\frac{N_c}{2p}C^p_{2p}\Lambda^{2p}\,,\qquad u^0_{2p+1}=0\,, \tag{3.4}$$

where the C_{2p}^p are the binomial coefficients.[1] The generalization of (3.4) to the case with matter has been addressed in [30, 32, 31], and the expressions for the moduli that factorize the curve (3.3) become very complicated compared to the pure gauge theory case. However, for the computation of the exact effective superpotential that we develop in this section, we just need the simple form of (3.4).

One can deform this $\mathcal{N}=2$ theory to a $\mathcal{N}=1$ gauge theory by adding a tree level superpotential

$$W_{\text{tree}} = \sum_{p\geq 1} \frac{g_p}{p} \text{tr}\phi^p \,. \tag{3.5}$$

The presence of this superpotential will lift the quantum moduli space, characteristic of the $\mathcal{N}=2$ Coulomb phase, except for the dimension 1 submanifolds, where N_c-1 mutually local magnetic monopoles become massless [22].

In order to compute the exact superpotential in this case it is useful to write eq. (2.14) in a slightly different way. Let us take the derivative with respect to S, so that we get

$$\frac{\partial}{\partial S}\, g_s \left\langle \text{tr} W(\Phi)\right\rangle_0 = \frac{\partial \mathcal{F}_{\chi=2}}{\partial S} - S\frac{\partial^2 \mathcal{F}_{\chi=2}}{\partial S^2} \,. \tag{3.6}$$

Now we have all the information that we need to compute $\mathcal{F}_{\chi=2}$ to all orders in perturbation theory. It was found in [1] that the full contribution coming from $\mathcal{F}_{\chi=2}$ to the effective superpotential is given by

$$W^0_{\text{eff}} = N_c\, \frac{\partial \mathcal{F}_{\chi=2}}{\partial S} = N_c\, S\left(-\log\frac{S}{\tilde{\Lambda}^3}+1\right) + N_c \frac{\partial \mathcal{F}^{pert}_{\chi=2}}{\partial S} \,, \tag{3.7}$$

where the first piece is the Veneciano-Yankielowitz superpotential for pure SU(N_c) super Yang-Mills [26]. Also $\mathcal{F}^{pert}_{\chi=2}$ is given by a perturbative expansion in S

$$\mathcal{F}^{pert}_{\chi=2} = \sum_{n\geq 1} f_n^{\chi=2}(g_p)\, S^{n+2} \,. \tag{3.8}$$

Inserting (3.7) and (3.8) in (3.6) we get

$$N_c\, \frac{\partial}{\partial S}\, g_s \left\langle \text{tr} W(\Phi)\right\rangle_0 = W^0_{\text{eff}} - S\frac{\partial W^0_{\text{eff}}}{\partial S} = N_c\, S - N_c \sum_{n\geq 1} n(n+2) f_n^{\chi=2}(g_p)\, S^{n+1} \,. \tag{3.9}$$

We should take now into account the following fact: when one considers a $\mathcal{N}=2$ theory, the moduli u_n are given by $u_n = \frac{1}{n}\text{tr}(\phi^n)$, where ϕ is the scalar component of the adjoint $\mathcal{N}=1$ chiral superfield of the $\mathcal{N}=2$ vector multiplet. In the Seiberg-Witten approach, the *vevs* of these operators may be written in terms of integrals over the cycles of the Seiberg-Witten curve. On the other hand, on the matrix model side the expression for the *vevs* of the moduli u_n was calculated in [20, 14], and for the case we are considering in this section they are given by

$$u^0_p = N_c \frac{\partial}{\partial S}\frac{g_s}{p}\langle \text{tr}(\Phi^p)\rangle_0 \,. \tag{3.10}$$

[1] The case of U(N_c) can be obtained from the case of SU(N_c) by shifting $x \to x - u_1/N$. This will induce a shift in the moduli u_p [17] that should be taken into account for the generalization of our results to the U(N_c) theory.

As in the $W_{\rm tree}=0$ case we should recover the $\mathcal{N}=2$ theory, so that the u_p in (3.10) should be the same ones as the u_p in (3.4). Therefore

$$\frac{\partial}{\partial S} g_s\, \langle {\rm tr}(\Phi^{2p})\rangle_0 = C^p_{2p}\Lambda^{2p} \tag{3.11}$$

Using (3.11) it is easy to see that at the critical point of the superpotential $\partial_S W^0_{\rm eff}=0$ we get from (3.9)

$$\sum_{p\geq 1} g_{2p} u^0_{2p} = W^0_{\rm eff}\,, \tag{3.12}$$

as implied by the Intriligator, Leigh and Seiberg linearity principle [27, 17]. Also from here we can extract the relation between the glueball superfield S and the scale of the theory Λ at the critical point.

$$S_0 = \frac{1}{N_c}\frac{\partial W^0_{\rm eff}}{\partial \log\Lambda^2} = \sum_{p\geq 1}\frac{1}{2} g_{2p} C^p_{2p}\Lambda^{2p}\,. \tag{3.13}$$

This is the same relation as found in [17]

Now at the critical point it is straight forward to obtain from (3.9) and (3.11) that

$$\sum_{p\geq 1}\frac{1}{2p} g_{2p} C^p_{2p}\Lambda^{2p} = S_0 - \sum_{n\geq 1} n(n+2) f^{\chi=2}_n(g_p)\, S_0^{n+1}\,. \tag{3.14}$$

Using (3.13) we are able to extract from (3.14) the coefficients $f^{\chi=2}_n(g_p)$ in a recursive way. Actually, we get the following expression for the $\mathcal{F}_{\chi=2}$ to the effective superpotential

$$\begin{aligned}
W^0_{\rm eff} &= N_c\frac{\partial\mathcal{F}_{\chi=2}}{\partial S} = N_c S\left(-\log\frac{S}{g_2\Lambda^2}+1\right) + N_c\sum_{n\geq 1}(n+2) f^{\chi=2}_n(g_p) S^{n+1}\,,\\
f^{\chi=2}_1 &= \frac{1}{2}\frac{g_4}{g_2^2}\,,\\
f^{\chi=2}_{n\geq 2} &= \frac{C^{n+1}_{2(n+1)}}{2(n+2)(n+1)}\frac{g_{2(n+1)}}{g_2^{n+1}} - \sum_{l=1}^{n-1}\frac{l(l+2)}{n(n+2)} f^{\chi=2}_l \sum_{\substack{p_1,\ldots,p_{l+1}=1\\ p_1+\cdots+p_{l+1}=n+1}}^{n+1}\frac{C^{p_1}_{2p_1} g_{2p_1}\cdots C^{p_{l+1}}_{2p_{l+1}} g_{2p_{l+1}}}{2^{l+1}\, g_2^{n+1}}\,.
\end{aligned} \tag{3.15}$$

It is very easy to check that the exact result for the effective superpotential obtained here, and given by eq. (3.15) coincides with the one obtained in [17]. One just has to substitute the expression for the exact effective superpotential in [17] in the eq. (3.7) and check that it is fulfilled once one takes into account eq. (3.11) and eq. (3.13). The main difference in both expressions for the exact superpotential is that in [17] the dependence of the superpotential in the glueball superfield S is given in an implicit way, whereas here we write that dependence in an explicit way. The results presented here also agree with the ones appearing in [28, 29] for the special cases of quadratic and quartic tree level superpotentials. The expression (3.15) is valid for an arbitrary tree level superpotential. The result in (3.15) is just one part of the effective superpotential. To get the full answer we need now to compute the matter contribution encoded in $\mathcal{F}_{\chi=1}$.

JHEP06(2004)051

3.2 The matter contribution

Following the same guidelines as the ones presented in the previous subsection for the computation of $W^0_{\rm eff}$, we can compute the exact form of the $\mathcal{F}_{\chi=1}$ contribution to the effective superpotential.

Expanding (2.17) around the critical point at $\Phi = 0$ we have that

$$\mathcal{F}_{\chi=1} = \sum_{f=1}^{N_f} \left(\langle \mathrm{tr} \log m_f \rangle_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\, m_f^k} \langle \mathrm{tr} \Phi^k \rangle_0 \right) . \tag{3.16}$$

If we now take the derivative of (3.16) with respect to the glueball superfield S we find that

$$\frac{\partial \mathcal{F}_{\chi=1}}{\partial S} = \sum_{f=1}^{N_f} \left(\log m_f - \sum_{k=1}^{\infty} \frac{C^k_{2k}}{2k} \frac{\Lambda^{2k}}{m_f^{2k}} \right) , \tag{3.17}$$

where we have used the relation (3.11). Therefore $\mathcal{F}_{\chi=1}$ is just given by

$$\mathcal{F}_{\chi=1} = \int \sum_{f=1}^{N_f} \left(\log m_f - \sum_{k=1}^{\infty} \frac{C^k_{2k}}{2k} \frac{\Lambda^{2k}}{m_f^{2k}} \right) \left(\frac{\partial S_0}{\partial \Lambda} \right) d\Lambda \, . \tag{3.18}$$

Then, using (3.13) we can perform the integral in (3.18) so that we get

$$\mathcal{F}_{\chi=1} = \sum_{f=1}^{N_f} \sum_{l \geq 1} \frac{1}{2} g_{2l} C^l_{2l} \Lambda^{2l} \log m_f - \sum_{f=1}^{N_f} \sum_{k,l \geq 1} \frac{l C^l_{2l} C^k_{2k} g_{2l}}{4k(k+l) m_f^{2k}} \Lambda^{2(k+l)} \, . \tag{3.19}$$

If now we want the explicit dependence in the glueball superfield S of $\mathcal{F}_{\chi=1}$ in (3.19) we can do it by using (3.13), so that again we can compute the coefficients $f^{\chi=1}_n(g_p, m_f)$ recursively

$$\mathcal{F}_{\chi=1} = S \sum_{f=1}^{N_f} \log m_f + \sum_{n \geq 1} f^{\chi=1}_n (g_p, m_f) S^{n+1} \, , \qquad f^{\chi=1}_1 = -\frac{1}{2} \sum_{f=1}^{N_f} \frac{1}{g_2 m_f^2} \, , \tag{3.20}$$

$$f^{\chi=1}_{n \geq 2} = -\frac{1}{g_2^{n+1}} \sum_{f=1}^{N_f} \left(\sum_{\substack{k,l=1 \\ k+l=n+1}}^{n} \frac{l C^l_{2l} C^k_{2k} g_{2l}}{4k(n+1) m_f^{2k}} + \sum_{q=1}^{n-1} \frac{f^{\chi=1}_q}{2^{q+1}} \sum_{\substack{p_1,\ldots,p_{q+1}=1 \\ p_1+\cdots+p_{q+1}=n+1}}^{n+1} C^{p_1}_{2p_1} g_{2p_1} \cdots C^{p_{q+1}}_{p_{q+1}} g_{2p_{q+1}} \right) .$$

Note that the form of the effective superpotential is additive with respect to inclusion of flavors.

Then the final expression for the exact effective superpotential, showing explicitly the dependence on the glueball superfield S for an arbitrary tree level superpotential, is given by

$$W_{\rm eff} = W^0_{\rm eff} + \mathcal{F}_{\chi=1} \, , \tag{3.21}$$

where $W^0_{\rm eff}$ is given in (3.15) and $\mathcal{F}_{\chi=1}$ in (3.20). The fact that in the results presented in [30] the dependence on the glueball superfield enters in a highly non-linear way in the

JHEP06(2004)051

equation for the effective superpotential, makes it very complicated to compare both results. Nevertheless, we can check our result by comparing it with the one obtained in [10] for the particular case of a quadratic tree level superpotential, $W_{\rm tree} = \frac{1}{2}{\rm tr}\Phi^2$. For this particular case we have that $g_2 = 1$, $g_n = 0$, $n \neq 2$. Then we just get from (3.15) and (3.20)

$$W_{\rm eff} = N_c\, S\left(-\log\frac{S}{\Lambda^2} + 1\right) + S\sum_{f=1}^{N_f}\log m_f - \sum_{f=1}^{N_f}\sum_{k,l\geq 1}\frac{(2k-1)!}{k!(k+1)!m_f^{2k}}S^{k+l}\,. \tag{3.22}$$

As it can be seen from (3.22) we get the same perturbative expansion as the one obtained in [10] by a different procedure (they sum over all the planar diagrams using the results in [34]).

It deserves to be emphasized that the results given in (3.15) and (3.20) gives us the exact effective superpotential for an arbitrary level superpotential in a remarkably simple way. Also note that the dependence of the effective superpotential on the glueball superfield is written explicitly.

4. Effective superpotentials in general vacua

In this section we will explain how to compute the effective superpotential using (2.14) and (2.17) at a general point of the moduli space of the gauge theory. Let us consider that we have a $\mathcal{N}=2$ supersymmetric gauge theory broken to $\mathcal{N}=1$ by the addition of a tree-level superpotential $W(\phi)$ to the gauge theory

$$W_{\rm tree} = \sum_{p=1}^{n+1} g_p\frac{{\rm tr}\phi^p}{p}\,. \tag{4.1}$$

Let us also consider that this tree level superpotential has n non coincident critical points. This will mean that we will have n glueball superfield S_i, $i = 1,\ldots,n$ (one at each critical point).

Now we want to use in this case the analysis developed in section 2 to compute the effective superpotential. For this purpose we have to take into account the fact that the formula (2.14) will now assume the following form

$$g_s\,\langle{\rm tr}W(\Phi)\rangle_0 = 2\mathcal{F}_{\chi=2} - \sum_{i=1}^{n} S_i\frac{\partial\mathcal{F}_{\chi=2}}{\partial S_i}\,, \tag{4.2}$$

according to the fact that that we have n distinct glueball superfields. On the other hand, the formula (2.17) remains unchanged as does not involve derivatives with respect to S.

In the case with n glueball superfields, the form of $\mathcal{F}_{\chi=2}$ around a critical point located at e_i can be written as

$$\mathcal{F}_{\chi=2} = \sum_{i=1}^{n} S_iW(e_i) - \frac{1}{2}\sum_{i=1}^{n} S_i^2\log\left(\frac{S_i}{W''(e_i)\Lambda^2}\right) - \sum_{i=1}^{N}\sum_{j\neq i} S_iS_j\log\left(\frac{e_i-e_j}{\Lambda}\right) + \sum_{p\geq 3}\mathcal{F}_p^{\chi=2} \tag{4.3}$$

where the Veneziano-Yankielowitz part of $\mathcal{F}_{\chi=2}$ was calculated in [20] from the matrix model integral. The coefficients $\mathcal{F}_m^{\chi=2}$ are polynomials of order p in S_i. Now introducing (4.3) into (4.2) we get

$$g_s\left\langle \mathrm{tr} W(\Phi)\right\rangle = \sum_{i=1}^{n} S_i W(e_i) + \frac{1}{2}\sum_{i=1}^{n} S_i^2 - \sum_{m\geq 3}(m-2)\mathcal{F}_m^{\chi=2}\,. \tag{4.4}$$

Also for the matter contribution we have

$$\mathcal{F}_{\chi=1} = \sum_{f=1}^{N_f}\sum_{i=1}^{n} S_i\,\log(e_i+m_k) + \sum_{q\geq 2}\mathcal{F}_q^{\chi=1} = g_s\sum_{f=1}^{N_f}\langle \mathrm{tr}\log(\Phi+m_f)\rangle\,, \tag{4.5}$$

where $\mathcal{F}_q^{\chi=1}$ are polynomials in S_k of order q. The coefficients $\mathcal{F}_m^{\chi=2}$ and $\mathcal{F}_q^{\chi=1}$ can be computed perturbatively within the matrix model. However we now explain how to compute those coefficients just by using the spectral curve associated with the matrix model.

Therefore, in order to compute the effective superpotential we need to compute the vacuum expectation values in (4.4) and (4.5). Within the matrix model framework these expectation values can be calculated easily following [7] by introducing the resolvent

$$\omega(x) \equiv g_s \mathrm{tr}\langle\frac{1}{x-\Phi}\rangle\,. \tag{4.6}$$

This resolvent is specified in terms of the loop equation

$$\omega(x)^2 = \omega(x)W'(x) + \frac{1}{4}f_{n-1}(x)\,, \tag{4.7}$$

where $f_{n-1}(z)$ is an arbitrary polynomial of the order $n-1$, and $W'(x) = \prod_{i=1}^{n}(x-e_i)$. From the equation (4.7) we read that the resolvent is given by

$$\omega(x) = \frac{1}{2}\left(W'(x) - \sqrt{W'(x)^2 + f_{n-1}(x)}\right)\,. \tag{4.8}$$

The values of the glueball superfields at the critical point e_i will be specified by the n coefficients of the polynomial $f_{n-1}(x)$ through the integrals [1]

$$S_i = \frac{1}{2\pi i}\oint_{A_i}\omega(x)dx\,, \tag{4.9}$$

where by A_i we denote a cycle enclosing the branch point centered in point e_i of the curve $y=\sqrt{W'(x)^2+f_{n-1}(x)}$, where y is the spectral curve associated with the matrix model.

Via the resolvent $\omega(x)$ one can easily calculate the expectation values of the single trace operators $\mathrm{tr}\Phi^k$ like [7]

$$g_s\langle \mathrm{tr}\Phi^k\rangle = \frac{1}{2\pi i}\oint_{A} x^k\omega(x)dx \tag{4.10}$$

where $A = \sum_{i=1}^{n} A_i$.

In order to being able to compute vacuum expectation values with the help of (4.10) we will use the parametrization of the polynomial f_{n+1} that appears in (4.7) given in [36], that we have found to be very useful, and that is given by

$$f_{n-1}(x) = \sum_{i=1}^{n} \tilde{S}_i \prod_{j\neq i}^{n} (x - e_j) = W'(x) \sum_{i=1}^{n} \frac{\tilde{S}_i}{x - e_i} \,. \tag{4.11}$$

Note that this polynomial is of degree $n-1$ only, as it should be. Now with the help of (4.11) we can compute S_i in terms of $\tilde{S}_k$ and e_j by computing the period integral (4.9). This period integral can be computed by reducing the evaluation of the integral to a set of residue calculations. Therefore if we expand the resolvent (4.7) around the point $\tilde{S}_k$, $k = 1, \dots, n$, the integral (4.9) over the cycle A_i can be performed just by calculating the residues at the point $x = e_i$

$$\begin{aligned} S_i &= \frac{1}{2\pi i} \oint_{A_i} \omega(x) dx \qquad (4.12)\\ &= \tilde{S}_i + \sum_{\substack{p=0\\ m\geq 2}}^{m} 2^{m-2} \frac{(2m-3)!!}{p!(m-p)!} \frac{\tilde{S}_i^p}{(m+p-2)!} \frac{\partial^{m+p-2}}{\partial x^{m+p-2}} \frac{1}{R_i(x)^{m-1}} \left(\sum_{j\neq i} \frac{\tilde{S}_j}{x - e_j} \right)^{m-p} \Bigg|_{x=e_i} , \end{aligned}$$

where by $R_i(x)$ we denote the polynomial $R_i(x) = \prod_{j\neq i}(x - e_j)$. Note that at first order in $\tilde{S}$ we have that $S_i = \tilde{S}_i$. This will be important later on. Also with the help of (4.11) and (4.10) we can compute $g_s \langle \mathrm{tr} W(\Phi) \rangle$ using the same procedure. We get that

$$\begin{aligned} g_s \langle \mathrm{tr} W(\Phi) \rangle &= \frac{1}{2\pi i} \oint_A W(x) \omega(x) dx = \sum_{i=1}^{n} W(e_i) \tilde{S}_i + \qquad (4.13)\\ &+ \sum_{\substack{p=0\\ m\geq 2}}^{m} 2^{m-2} \frac{(2m-3)!!}{p!(m-p)!} \frac{\tilde{S}_i^p}{(m+p-2)!} \frac{\partial^{m+p-2}}{\partial x^{m+p-2}} \frac{W(x)}{R_i(x)^{m-1}} \left(\sum_{j\neq i} \frac{\tilde{S}_j}{x - e_j} \right)^{m-p} \Bigg|_{x=e_i} . \end{aligned}$$

Now, due to the fact that at first order on $\tilde{S}$ we have that $S_i = \tilde{S}_i$, it is easy to see to use (4.12) to rewrite (4.13) in the form (4.4), and extract from there the form of the coefficients $\mathcal{F}_p^{\chi=2}$. For example for $\mathcal{F}_3^{\chi=2}$ we get

$$\mathcal{F}_3^{\chi=2} = \sum_{i=1}^{n} \frac{S_i^3}{R_i^2} \left(\frac{1}{4} R_i'' - \frac{2(R_i')^2}{3R_i} \right) + \frac{S_i^2}{R_i} \sum_{j\neq i} S_j \left(\frac{1}{e_{ij}^2} + \frac{2R_i'}{R_i e_{ij}} \right) - \frac{2S_i}{R_i} \sum_{j\neq i, k\neq i,j} \frac{S_j S_k}{e_{ij} e_{ik}} \,, \tag{4.14}$$

where $e_{ij} = e_i - e_j$ and $R_i = \prod_{j\neq i}(e_i - e_j)$, and we have used that $\partial^n W(e_i) = (n-1)\partial^{n-2} R_i(e_i)$. The result for $\mathcal{F}_3^{\chi=2}$ agrees with the one computed in [36] using Whitham hierarchies, and also with the one computed in [20] from a perturbative matrix model point of view. Also note that in the particular case where $S_{j\neq i} = 0$ (that is, at the maximally degenerating point) we recover the results computed in the previous section for the coefficient of order three in S.

The coefficients $\mathcal{F}_p^{\chi=2}$ give us one part of the effective superpotential. In order to get the matter contribution we need to compute (4.5). Following the same procedure as before

JHEP06(2004)051

we get that

$$g_s \sum_{f=1}^{N_f} \langle \mathrm{tr}\log(\Phi + m_f)\rangle = \sum_{f=1}^{N_f} \frac{1}{2\pi i}\oint_A \log(x+m_f)\omega(x)dx = \sum_{f=1}^{N_f}\sum_{i=1}^{n} S_i \log(e_i + m_f) +$$
$$+ \sum_{\substack{p=0 \\ m\geq 2}}^{m} 2^{m-2}\frac{(2m-3)!!}{p!(m-p)!}\frac{\tilde{S}_i^p}{(m+p-2)!}\frac{\partial^{m+p-2}}{\partial x^{m+p-2}} \times$$
$$\times \frac{\log(x+m_f)}{R_i(x)^{m-1}}\left(\sum_{j\neq i}\frac{\tilde{S}_j}{x-e_j}\right)^{m-p}\Bigg|_{x=e_i}. \tag{4.15}$$

Again with the help of (4.12) it is possible to rewrite (4.15) in the form (4.5), and extract the form of the coefficients $\mathcal{F}_q^{\chi=1}$. For example $\mathcal{F}_2^{\chi=1}$ and $\mathcal{F}_3^{\chi=1}$ are given by

$$\mathcal{F}_2^{\chi=1} = -\sum_{f=1}^{N_f}\sum_{i=1}^{n}\left(\frac{S_i^2}{2R_i e_{if}^2} + \frac{S_i^2 R_i'}{R_i^2 e_{if}}\right) + \sum_{f=1}^{N_f}\sum_{i=1}^{n}\frac{2S_i}{R_i e_{if}}\sum_{j\neq i}\frac{S_j}{e_{ij}},$$
$$\mathcal{F}_3^{\chi=1} = \sum_{f=1}^{N_f}\sum_{i=1}^{n}\frac{S_i^3}{R_i e_{if}}\left(\frac{5R_i'R_i''}{R_i^3} - \frac{6(R_i')^3}{R_i^4} - \frac{2R_i'''}{3R_i^2} - \frac{2(R_i')^2}{R_i^3 e_{if}} + \frac{R_i''}{2R_i^2 e_{if}} - \right.$$
$$\left. - \frac{4R_i'}{3R_i^2 e_{if}^2} + \frac{1}{2R_i e_{if}^3} - 2\sum_{j\neq i}\frac{1}{e_{ij}^3 R_j}\right) +$$
$$+ \frac{S_i^2}{R_i e_{if}}\left(\frac{8}{R_i e_{if}^2}\sum_{j\neq i}\frac{S_j}{e_{ij}} + \frac{4R_i'}{R_i^2 e_{if}}\sum_{j\neq i}\frac{S_j}{e_{ij}} + \frac{12(R_i')^2}{R_i^3}\sum_{j\neq i}\frac{S_j}{e_{ij}} - \frac{5R_i''}{R_i^2}\sum_{j\neq i}\frac{S_j}{e_{ij}} - \right.$$
$$\left. - \frac{4}{e_{if}^2}\sum_{j\neq i}\frac{S_j R_j'}{R_j e_{ij}^2} + \frac{1}{R_i e_{if}}\sum_{j\neq i}\frac{S_j}{e_{ij}^2} + \frac{8R_i'}{R_i^2}\sum_{j\neq i}\frac{S_j}{e_{ij}^2} + \frac{4}{e_{if}^2}\sum_{j\neq i}\frac{S_j}{R_j e_{ij}^3}\right) -$$
$$- \frac{S_i}{R_i e_{if}}\left(\frac{4}{R_i e_{if}}\sum_{\substack{j\neq i \\ k\neq j,i}}\frac{S_j S_k}{e_{ij}e_{ik}} + \frac{16R_i'}{R_i^2}\sum_{\substack{j\neq i \\ k\neq j,i}}\frac{S_j S_k}{e_{ij}e_{ik}} + \frac{8}{R_i}\sum_{\substack{j\neq i \\ k\neq j,i}}\frac{S_j S_k}{e_{ij}e_{ik}^2} + \right.$$
$$\left. + \frac{12}{R_i}\sum_{\substack{j\neq i \\ k\neq j,i}}\frac{S_j S_k}{e_{ij}^2 e_{ik}} - \sum_{\substack{j\neq i \\ k\neq j,i}}\frac{4R_j' S_j S_k}{R_j^2 e_{ij}e_{jk}} - \sum_{\substack{j\neq i \\ k\neq j,i}}\frac{4S_j S_k}{R_j e_{ij}e_{jk}^2}\right) \tag{4.16}$$

where $e_{if} = e_i - m_f$. The result for $\mathcal{F}_2^{\chi=1}$ agrees with the one computed in [20] perturbatively. As in the previous case, also note that if we set $S_{j\neq i} = 0$ we recover the results computed in the previous section for the coefficients of order two and three in S. Notice that in this general case, as well as in the confining vacua case, the dependence of $\mathcal{F}_{\chi=1}$ on the flavors is additive.

5. Conclusions

In this paper we have considered the Dijkgraaf-Vafa approach to $\mathcal{N}=1$ supersymmetric theories with $\mathrm{SU}(N_c)$ gauge group and $N_f < N_c$ matter hypermultiplets in the fundamental

JHEP06(2004)051

representation of the gauge group. Using this approach one can write the $\mathcal{F}_{\chi=2}$ and $\mathcal{F}_{\chi=1}$ contributions to the effective superpotential in terms of traces of certain matrix model operators (see eq. (2.14) and eq. (2.17)). We found that those traces should be actually computed in the pure gauge theory even to calculate the effective superpotential for theories with flavor. This remarkable fact allow us to compute the effective superpotential for theories with fundamental matter in a simple way without the need of performing a perturbative matrix model calculation, and just by computing quantities in the pure gauge theory.

Furthermore, we find that at the point of the moduli space where a maximal number of monopoles become massless eq. (2.14) and eq. (2.17) allow us to compute the exact effective superpotential for theories with fundamental matter in a simple way. This maximally degenerating point correspond to the case where the underlying Seiberg-Witten curve factorizes completely. As we already mentioned, the moduli that factorizes the Seiberg-Witten curve in the pure (no matter) case are well known and are given by the simple expression (3.4). Even though we are considering theories with fundamental matter, eq. (2.14) and (2.17) tell us that those moduli are the only ingredient that we need to compute the exact effective superpotential in this case. We find that the result for the exact superpotential in this case is remarkably simple.

We also considered the generalization of these results to an arbitrary point of the moduli space with n distinct glueball superfields. We found that eq. (2.14) and (2.17) also allow us to compute the effective superpotential for theories with fundamental matter in that case. To compute the superpotential we just needed the information about the spectral curve y associated with the matrix model, $y = \sqrt{W'(x)^2 + f_{n-1}(x)}$, and the techniques developed in [7] to compute traces within the matrix model setup. The method developed in this paper provides a powerful tool to compute effective superpotentials avoiding perturbative matrix model calculations.

JHEP06(2004)051

Acknowledgments

The author would like to thank S. Naculich and H. Schnitzer for very useful conversations and for the critical reading of the manuscript. This work is supported by the DOE under grant DE-FG02-92ER40706.

References

[1] F. Cachazo, K.A. Intriligator and C. Vafa, *A large-N duality via a geometric transition*, *Nucl. Phys.* **B 603** (2001) 3 [hep-th/0103067].

[2] F. Cachazo and C. Vafa, *$N = 1$ and $N = 2$ geometry from fluxes*, hep-th/0206017.

[3] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255].

[4] R. Dijkgraaf and C. Vafa, *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [hep-th/0207106].

[5] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, `hep-th/0208048`.

[6] R. Dijkgraaf, M.T. Grisaru, C.S. Lam, C. Vafa and D. Zanon, *Perturbative computation of glueball superpotentials*, *Phys. Lett.* **B 573** (2003) 138 [`hep-th/0211017`].

[7] F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, *Chiral rings and anomalies in supersymmetric gauge theory*, *J. High Energy Phys.* **12** (2002) 071 [`hep-th/0211170`].

[8] N. Seiberg, *Adding fundamental matter to 'Chiral rings and anomalies in supersymmetric gauge theory'*, *J. High Energy Phys.* **01** (2003) 061 [`hep-th/0212225`].

[9] F. Cachazo, N. Seiberg and E. Witten, *Phases of $N = 1$ supersymmetric gauge theories and matrices*, *J. High Energy Phys.* **02** (2003) 042 [`hep-th/0301006`]; *Chiral rings and phases of supersymmetric gauge theories*, *J. High Energy Phys.* **04** (2003) 018 [`hep-th/0303207`].

[10] R. Argurio, V.L. Campos, G. Ferretti and R. Heise, *Exact superpotentials for theories with flavors via a matrix integral*, *Phys. Rev.* **D 67** (2003) 065005 [`hep-th/0210291`].

[11] J. McGreevy, *Adding flavor to Dijkgraaf-Vafa*, *J. High Energy Phys.* **01** (2003) 047 [`hep-th/0211009`].

[12] I. Bena and R. Roiban, *Exact superpotentials in $N = 1$ theories with flavor and their matrix model formulation*, *Phys. Lett.* **B 555** (2003) 117 [`hep-th/0211075`].

[13] Y. Demasure and R.A. Janik, *Effective matter superpotentials from wishart random matrices*, *Phys. Lett.* **B 553** (2003) 105 [`hep-th/0211082`].

[14] S.G. Naculich, H.J. Schnitzer and N. Wyllard, *Matrix model approach to the $N = 2$ $\mathrm{U}(N)$ gauge theory with matter in the fundamental representation*, *J. High Energy Phys.* **01** (2003) 015 [`hep-th/0211254`].

[15] C. Hofman, *Super Yang-Mills with flavors from large-N_f matrix models*, *J. High Energy Phys.* **10** (2003) 022 [`hep-th/0212095`].

[16] R. Dijkgraaf, S. Gukov, V.A. Kazakov and C. Vafa, *Perturbative analysis of gauged matrix models*, *Phys. Rev.* **D 68** (2003) 045007 [`hep-th/0210238`].

[17] F. Ferrari, *On exact superpotentials in confining vacua*, *Nucl. Phys.* **B 648** (2003) 161 [`hep-th/0210135`].

[18] F. Ferrari, *Quantum parameter space and double scaling limits in $N = 1$ super Yang-Mills theory*, *Phys. Rev.* **D 67** (2003) 085013 [`hep-th/0211069`]; *Quantum parameter space in super Yang-Mills, II*, *Phys. Lett.* **B 557** (2003) 290 [`hep-th/0301157`].

[19] H. Itoyama and A. Morozov, *The Dijkgraaf-Vafa prepotential in the context of general Seiberg-Witten theory*, *Nucl. Phys.* **B 657** (2003) 53 [`hep-th/0211245`].

[20] S.G. Naculich, H.J. Schnitzer and N. Wyllard, *The $N = 2$ $\mathrm{U}(N)$ gauge theory prepotential and periods from a perturbative matrix model calculation*, *Nucl. Phys.* **B 651** (2003) 106 [`hep-th/0211123`].

[21] M. Gomez-Reino, S.G. Naculich and H.J. Schnitzer, *Improved matrix-model calculation of the $N = 2$ prepotential*, *J. High Energy Phys.* **04** (2004) 033 [`hep-th/0403129`].

[22] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [`hep-th/9407087`], erratum *ibid.* **B430** (1994) 485 [`hep-th/9407087`]; *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [`hep-th/9408099`].

[23] M.R. Douglas and S.H. Shenker, *Dynamics of* $\mathrm{SU}(N)$ *supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 271 [hep-th/9503163].

[24] A. Dymarsky and V. Pestun, *On the property of Cachazo-Intriligator-Vafa prepotential at the extremum of the superpotential*, *Phys. Rev.* **D 67** (2003) 125001 [hep-th/0301135].

[25] M. Matone, *Seiberg-Witten duality in Dijkgraaf-Vafa theory*, *Nucl. Phys.* **B 656** (2003) 78 [hep-th/0212253];
M. Matone and L. Mazzucato, *Branched matrix models and the scales of supersymmetric gauge theories*, *J. High Energy Phys.* **07** (2003) 015 [hep-th/0305225].

[26] G. Veneziano and S. Yankielowicz, *An effective lagrangian for the pure* $N = 1$ *supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 113** (1982) 231.

[27] K.A. Intriligator, R.G. Leigh and N. Seiberg, *Exact superpotentials in four-dimensions*, *Phys. Rev.* **D 50** (1994) 1092 [hep-th/9403198].

[28] H. Fuji and Y. Ookouchi, *Comments on effective superpotentials via matrix models*, *J. High Energy Phys.* **12** (2002) 067 [hep-th/0210148].

[29] G. Hailu and H. Georgi, *On exact superpotentials, free energies and matrix models*, *J. High Energy Phys.* **02** (2004) 038 [hep-th/0401101].

[30] Y. Demasure and R.A. Janik, *Explicit factorization of Seiberg-Witten curves with matter from random matrix models*, *Nucl. Phys.* **B 661** (2003) 153 [hep-th/0212212].

[31] R.A. Janik, *Exact* $\mathrm{U}(N_c) \to \mathrm{U}(N_1) \times \mathrm{U}(N_2)$ *factorization of Seiberg-Witten curves and* $N = 1$ *vacua*, *Phys. Rev.* **D 69** (2004) 085010 [hep-th/0311093].

[32] K.D. Kennaway and N.P. Warner, *Effective superpotentials, geometry and integrable systems*, hep-th/0312077.

[33] B.M. Gripaios, *Superpotentials for gauge and conformal supergravity backgrounds*, hep-th/0311025;
B.M. Gripaios and J. F. Wheater, *Veneziano-Yankielowicz superpotential terms in* $N = 1$ *SUSY gauge theories*, *Phys. Lett.* **B 587** (2004) 150 [hep-th/0307176].

[34] E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, *Planar diagrams*, *Commun. Math. Phys.* **59** (1978) 35.

[35] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, *Simple singularities and* $N = 2$ *supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 344** (1995) 169 [hep-th/9411048];
P.C. Argyres and A.E. Faraggi, *The vacuum structure and spectrum of* $N = 2$ *supersymmetric* $\mathrm{SU}(N)$ *gauge theory*, *Phys. Rev. Lett.* **74** (1995) 3931 [hep-th/9411057].

[36] H. Itoyama and A. Morozov, *Calculating gluino condensate prepotential*, *Prog. Theor. Phys.* **109** (2003) 433 [hep-th/0212032]; *Gluino-condensate (CIV-DV) prepotential from its Whitham-time derivatives*, *Int. J. Mod. Phys.* **A 18** (2003) 5889 [hep-th/0301136].